REMARKS ON THE CATEGORIFICATION OF COLORED JONES POLYNOMIALS

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Abstract. We introduce colored Jones polynomials of nanowords and their categorification. We also prove the existence of a Khovanov-type bicomplex which has three grades.

1. Colored Jones polynomials of nanowords and Khovanov homology

Jun Murakami introduced the notion of parallel versions of link invariants and gave a generalization of Jones polynomials, regarded as colored Jones polynomials nowadays, of links using cabling [Mu]. In this paper, the $(m_1, m_2, \ldots, m_l)$-cable of a l-component framed link diagram D is defined by replacing i-th component of D by m_i parallel strands, pushed off in the direction of the normal vector at every point. The $\mathbf{m}$-cable of a framed link L is given by taking each m_i-cable of the diagram of L using the blackboard framing. A colored Jones polynomial is one of the Reshetikhin–Turaev invariant [RT] for oriented framed links whose components are colored by irreducible representations of $\mathcal{U}_q(sl_2)$. The colored Jones polynomial $J_{\mathbf{n}}(L)$ by cabling for a l-component framed link L can be written as follows [KM] using $V_1 \otimes V_n \simeq V_{n+1} \oplus V_{n-1}$:

$$J_{\mathbf{n}}(L) = \sum_{\mathbf{k}=\mathbf{0}}^{\lfloor \frac{\mathbf{n}}{2} \rfloor} (-1)^{|\mathbf{k}|} \binom{\mathbf{n}-\mathbf{k}}{\mathbf{k}} J(L^{\mathbf{n}-2\mathbf{k}}) \tag{1}$$

where $\mathbf{n} = (n_1, n_2, \ldots, n_l)$, $\mathbf{k} = (k_1, k_2, \ldots, k_l)$ for arbitrary n_i, $k_i \in \mathbb{Z}_{\geq 0}$, $|\mathbf{k}| = \sum_i k_i$, $\binom{\mathbf{n}-\mathbf{k}}{\mathbf{k}} = \prod_{i=1}^{l} \binom{n_i - k_i}{k_i}$ and $J(L)$ is the Jones polynomial of L.

Here we define $\mathbf{m}$-cable of a nanophrases over an arbitrary α (cf. [G1]). Note that Vladimir Turaev defined cables of virtual strings [T1] and Andrew Gibson defined cables of nanowords corresponding to virtual strings [G1]. We also note that nanophrases are generalization of links and their theory were introduced by Turaev [T2]. Let P be a r-component nanophrase whose definition is in [GI]. We define the order of components by reading from the left to the right. We use the terms *single-component letter* and *two-component letter* defined as in [G2]. For a single-component letter A with $|A| = a$ in kth component of P, we define the word

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$w^j_{ki}(A)$ by

$$
(2) \qquad w^j_{ki}(A) = \begin{cases} A_{i0}A_{i1}A_{i2}\dots A_{ij-1} & A: \text{the first occurrence} \\ A_{ij-1}A_{ij-2}\dots A_{i2}A_{i1} & A: \text{the second occurrence} \end{cases}
$$

where $|A_{ij}| = a$ for every A_{ij}. For two-component letter A with $|A| = a$ in kth and lth components, let $w^j_{ki}(A)$ be A_i and $w^j_l(A)$ $A_0A_1\dots A_{j-1}$ where $|A_i| = a$ for every A_i. Using concatenations of words, we extend the definition of w^j_{ki} for letters to that of words, that is, $w^j_{ki}(w)$ consists of $w^j_{ki}(A)$ of single-component letters and two-component letters. For kth component w of a nanophrase P, the j-cable of w is defined by the replacing w with $w^j_{k0}(w)|w^j_{k1}(w)|\dots|w^j_{kj-1}(w)$. For the other components, every two-component letter A in kth and lth components is replaced by $w^j_l(A)$ where l depends on A ($1 \le l \le r$). Let $\mathbf{m}$ be a sequence of r nonnegative integers $(m_1, m_2, \dots, m_r)$. The $\mathbf{m}$-cable of a r-component nanophrase $P^{\mathbf{m}}$ is defined by such replacements r times from the first component to the rth component. In particular, a S_1-homotopy class of pseudolinks determines a S_1-homotopy class of the $\mathbf{m}$-cable of P. This is because there is the mapping $P_1 \mapsto \overline{P_1}$ which is the nanophrase over α_* with $\alpha_1 \to \alpha_*$; $-1 \mapsto a_-$ and $1 \mapsto a_+$. Let p be the natural projection nanophrases over α_* to α_1; a_+, $b_+ \mapsto 1$ and a_-, $b_- \mapsto -1$ [T2]. The $\mathbf{m}$-cable of $\overline{P_1}$ is denoted by $\overline{P_1}^{\mathbf{m}}$. This is well defined because $\overline{P_1}$ determines the pointed link diagram on the orientable surface with the minimal genus. By the construction, $P^{\mathbf{m}}_1$ is equal to $p(\overline{P_1}^{\mathbf{m}})$.

We use the definition of the cabling of nanophrases over α_1 and give the following results.

Theorem 1. *There exist a cohomology group $\mathcal{H}^{k,i,j}_{\mathbf{n}}(P)$ and a colored Jones polynomial $J_{\mathbf{n}}(P)$ which are S_1-homotopy invariants of an arbitrary pseudolink P satisfying*

$$
(3) \qquad J_{\mathbf{n}}(P) = \sum_j q^j \sum_{i,k} (-1)^{i+k} \mathrm{rk}\mathcal{H}^{k,i,j}_{\mathbf{n}}(P).
$$

Using $\mathcal{U}_L(P)$ introduced by [FI], Theorem 1 deduces the following.

Corollary 1. *There exist a cohomology group $\mathcal{H}^{k,i,j}_{\mathbf{n}}(\mathcal{U}_L(P))$ and a colored Jones polynomial $J_{\mathbf{n}}(\mathcal{U}_L(P))$ which are Δ_α-homotopy invariants of an arbitrary nanoword P over an arbitrary α satisfying*

$$
(4) \qquad J_{\mathbf{n}}(\mathcal{U}_L(P)) = \sum_j q^j \sum_{i,k} (-1)^{i+k} \mathrm{rk}\mathcal{H}^{k,i,j}_{\mathbf{n}}(\mathcal{U}_L(P)).
$$

We give a proof of Theorem 1 as follows.

Proof. The existence of $\mathcal{H}^{k,i,j}_n(P)$ and $J_{\mathbf{n}}(P)$ are given by that of the Jones polynomial $\hat{J}(P)$ and its Khovanov homology $KH^{i,j}$ of a pseudolink P in [FI]. Then it is sufficient to show S_1-homotopy invariance of $\hat{J}(P)$ and its $KH^{i,j}$ of a pseudolink P. But this invariance has been held by [FI] in the theory of pseudolinks, then the proof is completed. $\square$

Remark 1. The S_1-homotopy invariance of $\hat{J}(P)$ can be shown by using nanowords over α_* and virtual links [T2]. The S_1-homotopy invariance of $\mathcal{H}_{\mathbf{n}}(P)$ can be shown by Manturov's theory for the Khovanov homology of virtual links [Ma].

Remark 2. A virtual link vL can be regarded as a nanoword $\tilde{P}$ over α_* since there is the bijection ι from the set of all virtual links to the set of all S_*-homotopy classes of nanowords over α_* and for a pseudolink vL, there exists natural projection $p : \tilde{P} \mapsto P$ [T2]. Then, Theorem 1 implies the existence of a colored Jones polynomial of virtual links and its categorification as follows.

Corollary 2. *There exist a cohomology group $\mathcal{H}_{\mathbf{n}}(p(vL))$ and a colored Jones polynomial $J_{\mathbf{n}}(p(vL))$ are S_*-homotopy invariants of an arbitrary nanoword $\tilde{P}$ over an arbitrary α satisfying*

$$J_{\mathbf{n}}(p(vL)) = \sum_j q^j \sum_{i,k} (-1)^{i+k} \mathrm{rk} \mathcal{H}_{\mathbf{n}}^{k,i,j}(p(vL)). \tag{5}$$

Proof. For an arbitrary vL, take a nanoword $\tilde{P}$ over α_* corresponding to vL under the identification by the bijection ι. On the other hand, there exists a pseudolink P such that $p(\tilde{P}) = P$. Then we have $p(vL) = p(\tilde{P}) = P$. Now we have (5) by using (3). □

2. A bicomplex associated with the colored Jones polynomial

In this section, we use the same notation as [I2]. We state the main result in this section. Let L be a framed (classical) link and D its diagram.

Theorem 2. *There exists a bicomplex $\{\mathcal{C}_{\mathbf{n}}^{k,i,j}(D), d'^{k,i,j}, d''^{k,i,j}\}$ satisfying*

$$J_{\mathbf{n}}(L) = \sum_j q^j \sum_{k,i} (-1)^{i+k} \mathrm{rk} \mathcal{H}^k(\mathcal{H}^i(\mathcal{C}_{\mathbf{n}}^{*,*,j}(D), d''^{k,i,j}), d'^{k,i,j}) \tag{6}$$

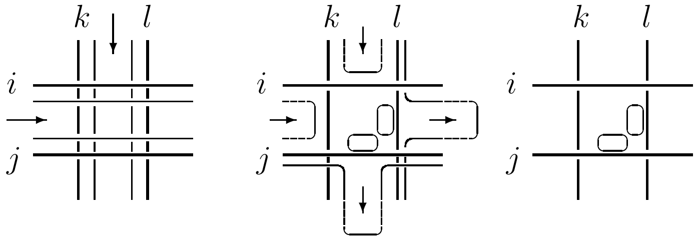
and $\mathcal{H}^k(\mathcal{H}^i(\mathcal{C}_{\mathbf{n}}^{*,*,j}(D), d''^{k,i,j}), d'^{k,i,j})$ is an invariant of a link L.

Proof. For the latter part of the statement, the invariance of the Khovanov homology $\mathcal{H}^i(\mathcal{C}_{\mathbf{n}}^{k,*,j}(D), d''^{k,i,j})$ is given by the retractions and chain homotopy maps of [I1]. For (6), if a bi-graded complex $\mathcal{C}_{\mathbf{n}}^{k,i,j}(D)$ has two differentials, (6) is held. Then it is sufficient to show that such a complex $\mathcal{C}_{\mathbf{n}}^{k,i,j}(D)$ with some differentials become a bicomplex.

Except for the last paragraph of the proof of Main Theorem in [I2], it is usable for this proof. Then let us assume that we are familiar with this part of the proof. Therefore we already know the definition of Type 1 that is an enhanced Kauffman state. We set $\delta_{s,t} = \delta_{0,0}$ and assume that T_1 stands for an enhanced Kauffman state given by removing all of the contracted strands from Type 1. It is necessary to select good choice of signs for Type 1 to redefine $d_{\mathbf{n}}'^{k,i,j}$.

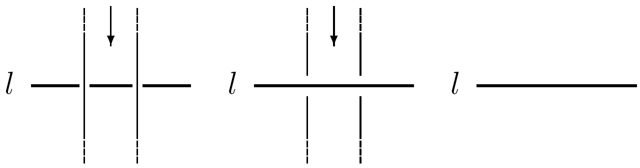
First we will show that "the number of contracted circles are even $(*)$" in the case of knot diagrams. Every set of crossings of the cable diagram can be represents as (a) of Figure 1 and its mirror image. If we take Type 1 of the diagram (a), then we get (b) of Figure 1 corresponding to (a) of Figure 1. For (a) of Figure 1, we consider four pairs of neighbor crossings which connecting (a), (b) needs to connect

at least two simple curves in the direction to the right and the bottom in Figure (b). Similarly, each of the other two simple curves connects other set of crossing alike (b). Every such connecting simple curve is denoted by two dashed lines and one edge (b). By shortening dashed line, Type 1 consists of (c)-type sets of crossings. This conclude $(*)$ in the case of knot diagrams. Now we consider the case of link diagrams. In the case of a link diagram, the set of crossings as (a) or (b) of Figure 2 arises. But, they do not produce any contracted circles since this set of crossings become (c) of Figure 2 when we take Type 1. Therefore, $(*)$ is still held in the case of link diagrams.



(a) (b) (c)

FIGURE 1. The thin lines denote 1-cable strands. The numbers i, j, k and l (≥ 0) with thick lines denote the number of parallel strands. For example, (a) has $i+1$, $j+1$, $k+1$ and $l+1$ strands. Circles in (b) and (c) stand for contracted circles.



(a) (b) (c)

FIGURE 2. (a), (b): Contracted strands encountered with l parallel non-contracted strands. (c): Type 1 corresponding to (a) or (b).

Second, we redefine signs of contracted circles of Type 1 preserving j-grade. For every part of Type 1 as (b) of Figure 1, we associate $+$ (resp. $-$) with the bottom (resp. right) circle in the rectangle of (c) of Figure 1.

Third, we define Type 2 for Type 1. Type 2 is an enhanced state by deleting two contracted circles in the rectangle of (c) of Figure 1. For (a) of Figure 2, it is sufficient to consider the case ┼┼ since the discussion of signs in this case ┼┼ is the same as that of the former case up to mirror symmetry. Let us look Figure 3. If (a) generates two circles of Type 1, Type 2 is defined by (b). If (c) generates only one circle of Type 1, Type 2 is defined by (d) where p and q are signs.

Finally, we define $d_{\mathbf{n}}^{k,i,j} : \mathcal{C}_{\mathbf{n}}^{k,i,j}(D) = \oplus_{\mathbf{s}\in I_{\mathbf{k}},|\mathbf{k}|=k}\mathcal{C}^{i,j}(D^{\mathbf{s}}) \to \oplus_{\mathbf{s}'\in I_{\mathbf{k}},|\mathbf{k}|=k+1}C^{i,j}(D^{s'})$ $= \mathcal{C}^{k+1,i,j}(D)$ by $\mathcal{C}^{i,j}(D^{\mathbf{s}}) \ni S\otimes[xy] \mapsto S'\otimes[x] \in \mathcal{C}^{i,j}(D^{\mathbf{s}'})$ where S' is T_1 (resp. $-T_1$) if S is Type 1 (resp. Type 2) 0 otherwise. On the other hand, for a $\mathbf{k}$-pairing $\mathbf{s}$, the differential $\delta_{0,0} : \mathcal{C}^{i,j}(D^{\mathbf{s}}) \to \mathcal{C}(D^{\mathbf{s}})$ of the original Khovanov homology $\mathcal{H}^{i,j}(D)$

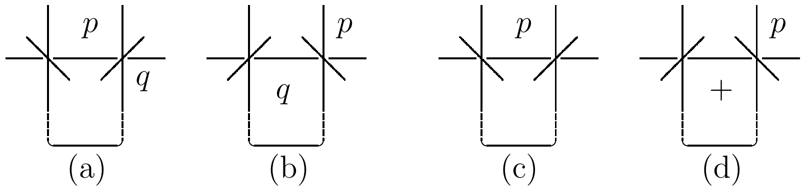


Figure 3. (a):A part of Type 1 generating two circles. (b): The part of Type 2 corresponding to (a). (c) A part of Type 1 generating only one circles. (d): The part of Type 1 corresponding to (c). In these figure, p and q stand for signs of enhanced Kauffman states.

$= H^i(C^{*,j}(D), \delta_{0,0})$ is denoted by $d_{\mathbf{s}}^{i,j}$. By this definition, we have $d_{\mathbf{s}'}^{i,j} \circ d_{\mathbf{n}}^{k,i,j} = d_{\mathbf{n}}^{k,i+1,j} \circ d_{\mathbf{s}}^{i,j}$. Setting $d'^{k,i,j} := (-1)^{(\mathbf{s},\mathbf{s}')} d_{\mathbf{n}}^{k,i,j}$ and $d''^{k,i,j} := (-1)^k \oplus_{\mathbf{s} \in I_{\mathbf{k}}, |\mathbf{k}|=k} d_{\mathbf{s}}^{i,j}$, we have $d''^{k+1,i,j} \circ d'^{k,i,j} + d'^{k,i+1} \circ d''^{k,i,j} = 0$. This completes the proof of Theorem 2. $\square$

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